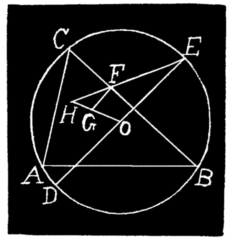
I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let O and H be the circum and ortho-centers respectively of the triangle ABC. Draw the diameter DE, connect E and H, and from F the mid-point of EH draw FG parallel to OE.



Now H and O are inverse points.

G is the mid-point of HO and $GF=\frac{1}{2}OE=$a constant.

$\therefore$ G is the center and GF the radius of the nine-point circle.

$\therefore$ The locus of F is the nine-point circle.

II. Solution by the PROPOSER.

Let $l\alpha+m\beta+n\gamma=0$..........(1) be any diameter. The isogonal transformation of (1) is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now (1), passing through the center of the circumcircle, the coordinates of which are proportional to $\cos A$, $\cos B$, $\cos C$, gives the relation

$$l\cos A+m\cos B+n\cos C=0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also, the center of (2), which is an equilateral hyperbola, with condition (3), is given by

$$\frac{l}{n}=\frac{-a\alpha^2+b\alpha\beta+c\alpha\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma},\quad \frac{m}{n}=\frac{a\alpha\beta-b\beta^2+c\beta\gamma}{b\beta\gamma-c\gamma^2+a\alpha\gamma} \ldots\ldots (4).$$

Dividing (3) by n, and substituting equations (4), and reducing,

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta-a\alpha^2\cos A-b\beta^2\cos B-c\gamma^2\cos C=0 \ldots\ldots (5),$$

the nine-points circle.

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line, may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

If the four points be A, B, C, D, and the axis of x coincide with the given straight line, A, B may be supposed given by

$$\alpha x^2+2\beta x+\gamma=0\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{or } x=\frac{-\beta\pm\sqrt{\beta^2-\gamma^2}}{\alpha}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\text{and } C,\ D, \text{ by } \alpha' x^2+2\beta' x+\gamma'=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

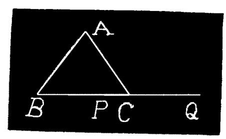
Now as long as γ exceeds β, (2) gives imaginary values for x, and so for a like pair of values for (3), which does not violate the condition

$$\alpha\gamma'+\alpha'\gamma=2\beta\beta'\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

any number of values of β, γ in (2) always being consistent with (4).

II. Solution by JOHN B. FAUGHT, A. M., Instructor in Mathematics, Indiana University, Bloomington, Indiana.

Using trilinear coordinates, take B and C for the two real points on the real line $\alpha=0$, i. e., $b\beta+c\gamma=2\triangle$. $B^2+K^2\gamma^2=0$, is the equation of two lines through A; that is $\beta+Ki\gamma=0$, and $\beta-Ki\gamma=0$. These lines form with $\beta=0$ (AC) and $\gamma=0$ (AB) a harmonic pencil, and hence intersect BC in two points forming with B and C a harmonic range.



Moreover these lines are imaginary for all real values of K and hence must intersect BC in imaginary points, otherwise they would contain two real points, which is impossible.

The coordinates of the points of intersections of these imaginary lines may be found by solving with $b\beta+c\gamma=2\triangle$. Thus $\beta=-Ki\gamma$ gives

$$(c-bKi)\gamma=2\triangle \text{ and } \gamma=\frac{2\triangle c}{c^2+b^2K^2}+\frac{2\triangle Kb}{c^2+b^2K^2}i$$

$$\text{and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}-\frac{2\triangle Kc}{c^2+b^2K^2}i, \text{ and } \beta=Ki\gamma, \text{ gives}$$

$$\gamma=\frac{2\triangle c}{c^2+b^2K^2}-\frac{2\triangle Kb}{c^2+b^2K^2}i, \text{ and } \beta=\frac{2\triangle K^2b}{c^2+b^2K^2}+\frac{2\triangle Kc}{c^2+b^2K^2}i.$$

If P and Q denote the imaginary points of intersection, we see that their coordinates are conjugates. These points are called "conjugués harmoniques" with respect to B and C, by M. Chasles.

It is evident that by giving different values to K an infinite number of such points can be found.

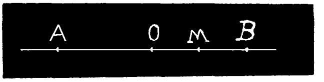
III. **Solution by the PROPOSER.**

The roots of $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$ will be harmonic if $ac'+a'c-2bb'=0$ (see Scott's Geometry, page 45).

Let $x^2=p^2$ give the points A and B. Let $x=OM=K<(OB=p)$ be midway between the other points, P and Q. The equation giving P and Q is

$$a'x^2+2b'x+c'=0, \text{ with the conditions } \frac{b'}{a}=-K, \text{ and } c'-p^2a'=0,$$

$$\text{or } x^2-2Kx+p^2=0.$$



But since $K<p$, $K^2-p^2<0$, the roots of this equation are imaginary, and since there are an indefinite number of values for $K<p$, there will be an indefinite number of pairs of imaginary points on the line harmonic with the given real pair. (Scott's Geometry, page 45.)

Solved in a similar manner by *G. B. M. ZERR.*

PROBLEMS.

63. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.**

Prove, analytically :—A rectangular hyperbola cannot be cut from a right circular cone unless the angle at its vertex is greater than a right angle.

64. **Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society and Treasurer of Grant County, Marion, Indiana.**

Let the bisectors of the angles A, B, C of a triangle meet the sides opposite A, B, C in A', B', C'. Let AA', BB', CC' meet the sides of the triangle $A'B'C'$ in A'', B'', C''. Let this process continue indefinitely. Express the sides and angles of the triangle $A^{(m)}B^{(m)}C^{(m)}$ in terms of the sides and angles of the original triangle ABC.